

EQUIPARTITION OF SEVERAL MEASURES

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ABSTRACT. We prove several results of the following type: any d measures in \mathbb{R}^d can be partitioned simultaneously into k equal parts by a convex partition (this particular result is proved independently by P. Soberón). Another example is: any convex body in the plane can be partitioned into q parts of equal areas and perimeters provided q is a prime power.

The above results give a partial answer to several questions posed by A. Kaneko, M. Kano, R. Nandakumar, N. Ramana Rao, and I. Bárány. The proofs in this paper are inspired by the generalization of the Borsuk–Ulam theorem by M. Gromov and Y. Memarian.

1. INTRODUCTION

We use the idea in Gromov’s generalization of the Borsuk–Ulam theorem from [12, 20] to prove the theorem on simultaneous partitioning of several measures into equal parts. More generally, we partition several measures into equal parts and require several continuous (in a certain sense) functions of these parts coincide.

Let us make some definitions. Consider a compact topological space X with a Borel probability measure μ . Let $C(X)$ denote the set of continuous functions on X .

Definition 1.1. A finite-dimensional linear subspace $L \subset C(X)$ is called *measure separating*, if for any $f \neq g \in L$ the measure of the set

$$e(f, g) = \{x \in X : f(x) = g(x)\}$$

is zero.

In particular, if X is a compact subset of \mathbb{R}^n such that $X = \text{cl}(\text{int } X)$ and μ is any absolutely continuous measure then any finite-dimensional space of analytic functions is measure-separating, because the sets $e(f, g)$ always have dimension $< n$ and therefore measure zero. Then for any collection of q elements of a measure-separating subspace we define a partition of X .

Definition 1.2. Suppose $F = \{u_1, \dots, u_q\} \subset C(X)$ is a family of functions such that $\mu(e(u_i, u_j)) = 0$ for all $i \neq j$. The sets (some of them may be empty)

$$V_i = \{x \in X : \forall j \neq i \ u_i(x) \geq u_j(x)\}$$

have a zero measure overlap, so they define a partition $P(F)$ of X . In case u_i are linear functions on \mathbb{R}^n we call $P(F)$ a *generalized Voronoi partition*.

Now we are ready to state the result about partitioning a measure into equal parts.

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Theorem 1.3. *Suppose L is a measure-separating subspace of $C(X)$ of dimension $n + 1$, μ_1, \dots, μ_n are absolutely continuous (with respect to the original measure on X) probability measures on X . Then for any prime power q there exists a q -element subset $F \subset L$ such that for every $i = 1, \dots, n$ the partition $P(F)$ partitions the measure μ_i into q equal parts.*

The “ham sandwich” theorem [24, 23] follows from this theorem, if we let $X = \mathbb{R}^n$, L be the space of polynomials of degree ≤ 1 , and $q = 2$. Moreover, taking $q > 2$ in this theorem we obtain partitions of \mathbb{R}^n into q convex parts, partitioning every measure μ_1, \dots, μ_n into q equal parts. This is true for prime powers q , and for arbitrary q it can be obtained by iterating partitions, though the partitions will no more be generalized Voronoi partitions. Such results about convex partitions of the plane were known, see [17] for example.

Some results similar to Theorem 1.3 were independently obtained in [5, 22], in those papers q was prime and the functions were linear (so the partition was convex). After the discussions between the author and the authors of [5] the second version of [5] was updated to include the prime power case. The reader may also find in the second version of [5] an analogue of Theorems 1.4 and 1.6 along with a detailed and rigorous discussion of all the continuity issues in those theorems and a detailed proof of Lemma 2.1, which is the main tool to obtain all these results. In the other paper [22] the proof uses much simpler topology (without any analogue of Lemma 2.1), which is still enough to prove Theorem 1.3 for any number of parts q .

Of course, in order to apply Theorem 1.3 to measures in \mathbb{R}^d we have to first suppose that the measures have compact support; for arbitrary probability measures we may use approximation and carefully go to the limit. The partition into convex parts is still possible, but possibly it will not be described as $P(F)$ for some system of linear functions F , that is it may not be a generalized Voronoi partition.

Let us give more results about generalized Voronoi partitions.

Theorem 1.4. *Suppose $C \subset \mathbb{R}^n$ is a convex body, μ is an absolutely continuous probability measure on C , $\varphi_1, \dots, \varphi_{n-1}$ are functions of convex compact sets continuous with respect to the Hausdorff metric, and q is a prime power. Then C can be partitioned into q convex parts V_1, \dots, V_q so that*

$$\mu(V_1) = \dots = \mu(V_q),$$

and for every $i = 1, \dots, n - 1$

$$\varphi_i(V_1) = \dots = \varphi_i(V_q).$$

In particular, we can take $\varphi_i(K)$ to be i -th Steiner measure, i.e. the coefficient at t^i in the polynomial (B is the unit ball here)

$$P_K(t) = \mu(K + tB).$$

As a particular case, we obtain the following fact: any compact convex set $C \subset \mathbb{R}^2$ can be partitioned into q convex parts with equal areas and perimeters, provided q is a prime power. Such results were conjectured and proved in particular cases $q = 3, 4$ in [21, 6], the smallest remaining open case in this question is therefore $q = 6$.

A similar theorem holds for the standard n -dimensional sphere and its convex subsets:

Theorem 1.5. *Suppose $C \subset S^n$ is a convex body, μ is an absolutely continuous probability measure on C , $\varphi_1, \dots, \varphi_{n-1}$ are functions of convex compact subsets of S^n continuous with respect to the Hausdorff metric, and $q > 1$ is a prime power. Then C can be partitioned into q convex parts V_1, \dots, V_q so that*

$$\mu(V_1) = \dots = \mu(V_q),$$

and for every $i = 1, \dots, n - 1$

$$\varphi_i(V_1) = \dots = \varphi_i(V_q).$$

The next theorem does not follow directly from Theorem 1.4 because of some discontinuity issues, but is proved in a similar manner. This is a higher-dimensional generalization of the results about perfect partitions in the plane, see [2].

Theorem 1.6. *Suppose $C \subset \mathbb{R}^n$ is a convex body, and for some $1 \leq k \leq n$ we have k absolutely continuous probability measures μ_1, \dots, μ_k on C , and $n - k$ absolutely continuous probability measures $\sigma_1, \dots, \sigma_{n-k}$ on ∂C . Then for any $q \geq 1$ the body C can be partitioned into q convex parts V_1, \dots, V_q so that for any $i = 1, \dots, k$*

$$\mu_i(V_1) = \dots = \mu_i(V_q),$$

and for every $i = 1, \dots, n - k$

$$\sigma_i(V_1 \cap \partial C) = \dots = \sigma_i(V_q \cap \partial C).$$

It would be interesting to generalize the above theorems in the following direction. Let us prescribe positive reals $\alpha_1, \dots, \alpha_q$ with $\alpha_1 + \dots + \alpha_q = 1$, several probability measures μ_1, \dots, μ_k and try to find a convex partition V_1, \dots, V_q of \mathbb{R}^n so that for every $1 \leq i \leq k$ and $1 \leq j \leq q$

$$\mu_i(V_j) = \alpha_j.$$

In [4] (and reproved in [27, 15]) such a result was established for $k = 1$ and any q . The partition had the form $P(F)$, where F is a set of linear functions with prescribed and distinct degree 1 homogeneous parts (and variable free terms). In [14] a similar result was established for two measures in \mathbb{R}^2 of special kind, the first being the standard area in a convex body K and the second being the length measure on ∂K . It seems that for $d \geq 3$ and $k \geq 2$ a convex partition is not sufficient and it makes sense to consider non-convex partitions. In [25, 3] it was shown that in the one-dimensional case it is enough to consider partitions into unions of segments with complexity bounded by q and k (approximately by the product of q and k), see also Section 7.

Using the same technique we slightly generalize the Borsuk–Ulam type theorem of Gromov and Memarian [20, Theorem 3]. The difference is that q is not required to be a power of two, but can be any prime power, and we partition several measures into equal parts at the same time.

First, we have to define a general notion of a center function.

Definition 1.7. Let $L \subset C(X)$ be a finite-dimensional linear subspace of functions. Suppose that for any subset $F \subset L$ such that all sets $\{V_1, \dots, V_q\} = P(F)$ have nonempty interiors we can assign *centers* $c(V_1), \dots, c(V_q) \in X$ to the sets. If this assignment is continuous w.r.t. F and equivariant (with respect to the permutations of functions in F and permutations of points in the sequence c_1, \dots, c_q), we call $c(\cdot)$ a *q -admissible center function* for L .

Theorem 1.8. *Suppose L is a measure-separating subspace of $C(X)$ of dimension $n + 1$, μ_1, \dots, μ_{n-k} ($n > k$) are absolutely continuous (with respect to the original measure on X) probability measures on X , $c(\cdot)$ is a q -admissible center function for some prime power q , and*

$$h : X \rightarrow \mathbb{R}^k$$

is a continuous map. Then there exists a q -element subset $F \subset L$ such that for every $i = 1, \dots, n - k$ the partition $P(F)$ partitions the measure μ_i into q equal parts, and we have

$$h(c(V_1)) = h(c(V_2)) = \dots = h(c(V_q))$$

for $\{V_1, \dots, V_q\} = P(F)$.

Remark 1.9. In the case $q = 2^\ell$ in all the above theorems, when we want a convex partition, the partition may be chosen to be a binary space partition by hyperplanes. In this case instead of using the configuration space $F_q(L)$ in Lemma 2.1 we may follow [20]: Take the first measure μ_1 and parameterize the binary equipartitions of μ_1 by the product of spheres $Q_q(\mathbb{R}^n) = (S^{n-1})^{\times q-1}$ taking the normals to the partitioning hyperplanes. The space $Q_q(\mathbb{R}^n)$ is equal to $\widetilde{M}(n, \ell)$ in the notation of [13, Definition 1.1]. Then note that partitioning of the remaining measures (or functions) into equal parts is guaranteed by non-vanishing of the Euler class (see the definitions in Section 3) $e(\alpha_q)^{n-1}$ in the cohomology $H_{\Sigma_q^{(2)}}^{(q-1)(n-1)}(Q_q(\mathbb{R}^n); \mathbb{F}_2)$ (the group $\Sigma_q^{(2)}$ is the 2-Sylow subgroup of the permutation group Σ_q). The latter fact is an analogue of Lemma 2.1, but its proof is much simpler: It is sufficient to construct a $\Sigma_q^{(2)}$ -equivariant map $Q_q(\mathbb{R}^n) \rightarrow \alpha_q^{n-1}$ (the *test map*) with a unique non-degenerate $\Sigma_q^{(2)}$ -orbit of zeros. This test map is actually defined in (3.1–3.3), if we assume the standard $\Sigma_q^{(2)}$ -equivariant inclusion $Q_q(\mathbb{R}^n) \subset F_q(\mathbb{R}^n)$ (as in [13, Lemma 1.6]).

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2. PROOF OF THEOREM 1.3

The set of all ordered q -tuples $F \subset L$ (collections of q pairwise distinct functions) is the configuration space $F_q(L)$, it has the natural action of the symmetric group Σ_q . Denote α_q the $(q-1)$ -dimensional representation of Σ_q , this is the subspace of vectors in \mathbb{R}^q with zero coordinate sum with the action of Σ_q by permuting the coordinates.

For every $i = 1, \dots, n$ and $P(F) = \{V_1, \dots, V_q\}$ the values

$$\mu_i(V_1) - \frac{1}{q}, \dots, \mu_i(V_q) - \frac{1}{q}$$

define a map $f_i : F_q(L) \rightarrow \alpha_q$, this map is Σ_q -equivariant, and from the absolute continuity and the measure separation property we deduce that the map f_i is continuous.

To prove the theorem we have to show that the direct sum map

$$f = f_1 \oplus \dots \oplus f_n : F_q(L) \rightarrow \alpha_q^n$$

maps some configuration F to zero.

The representation α_q defines a vector bundle $\alpha_q \times_{\Sigma_q} E\Sigma_q \rightarrow B\Sigma_q$ with the orientation sheaf $\pm\mathbb{Z} \times_{\Sigma_q} E\Sigma_q$ (here $\pm\mathbb{Z}$ is the Σ_q -module with the permutation sign action). So it makes sense to consider its Euler class in the cohomology $H^{q-1}(\Sigma_q; \pm\mathbb{Z})$, which in turn has a natural image in every $H_{\Sigma_q}^{q-1}(X; \pm\mathbb{Z})$ for every Σ_q -space X . Now we use the following Lemma [16, Lemma 6] (see also Section 3):

Lemma 2.1. *The image of $e(\alpha_q)^n$ is nonzero in the cohomology*

$$H_{\Sigma_q}^{(q-1)n}(F_q(\mathbb{R}^{n+1}); (\pm\mathbb{Z})^{\otimes n}) = H_{\Sigma_q}^{(q-1)n}(F_q(L); (\pm\mathbb{Z})^{\otimes n}).$$

Remark 2.2. In the above lemma we may reduce the cohomology coefficients mod p , where $q = p^\ell$.

The nonzero image of $e(\alpha_q)^n$ is naturally interpreted as the nonzero Euler class of the Σ_q -equivariant vector bundle

$$\eta : \alpha_q^n \times F_q(L) \rightarrow F_q(L),$$

the map f can be interpreted as a section of η , so it must have a zero.

3. PROOF OF LEMMA 2.1

Answering the remarks from Pavle Blagojević (private communication) and the unknown referee, we provide a proof of Lemma 2.1. Put $d = n + 1$ in this section.

In fact, most important cases of this lemma were previously known. For $q = p$ (i. e. a prime number) this lemma is valid even in \mathbb{Z}_p -equivariant cohomology (if we embed $\mathbb{Z}_p \subset \Sigma_p$ in the natural way). This is a particular case of [16, Lemma 5], and seems to be known much before, see [10, Theorem 3.4, Corollaries 3.5 and 3.6] for example. The case $q = 2^\ell$ follows from the direct computations in [13], reproduced implicitly in [20] (see also Remark 1.9), in this case it holds in $\Sigma_q^{(2)}$ -equivariant cohomology mod 2.

Lemma 2.1 for $d = 2$ was actually proved in [26]. Below we reproduce the proof extended to all $d \geq 2$. We denote $F_q(\mathbb{R}^d)/\Sigma_q$ by $B_q(\mathbb{R}^d)$ and denote the natural projection $F_q(\mathbb{R}^d) \rightarrow B_q(\mathbb{R}^d)$ by π . While this paper was under review, another proof of this lemma (also following [26]) appeared in the second version of [5]. So the reader may find more details and explanations in [5].

Let us introduce one important construction: by projecting the configuration of q points $F = \{p_1, \dots, p_q\}$ in \mathbb{R}^d onto the coordinate axes x_2, \dots, x_d we obtain the average for every $j = 2, \dots, q$

$$(3.1) \quad m_j(F) = \frac{1}{q}(x_j(p_1) + \dots + x_j(p_q))$$

and q numbers

$$(3.2) \quad x_j(p_1) - m_j(F), \dots, x_j(p_q) - m_j(F),$$

which constitute a Σ_q -equivariant map $h_j : F_q(\mathbb{R}^d) \rightarrow \alpha_q$. Totally these maps constitute a map

$$(3.3) \quad \tilde{h} : F_q(\mathbb{R}^d) \rightarrow \alpha_q^{d-1}$$

with the zero set \tilde{Z} consisting of configurations $F = \{p_1, \dots, p_q\}$ such that $x_j(p_i)$ does not depend on i for $j \geq 2$. In other words, the set \tilde{Z} consists of configurations with all points lying on a single line parallel to the first coordinate axis.

Now let us remind the notion of the Fuks cellular partition of $F_q(\mathbb{R}^d)$ [11]. Consider an oriented graded tree T of height d (levels are numbered from 1 to $d + 1$ from leaves to the root) with q leaves labeled by numbers $i = 1, \dots, q$. If a vertex v on level $j + 1$ of this tree has children w_1, \dots, w_k (in this order) on level j then for every pair of children w_a, w_b with $a < b$ and every labels i_a and i_b on a descendant of w_a and a descendant of w_b respectively we impose the inequality $x_j(p_{i_a}) < x_j(p_{i_b})$ on the coordinates of the configuration F . Note that these inequalities together guarantee that the points p_i are pairwise distinct and so to any tree T we associate an open cell $Z_T \subset F_q(\mathbb{R}^d)$. If we remove labels on the bottom level then we obtain an open cell $\pi(Z_T)$ of $B_q(\mathbb{R}^d)$. It is easy to note that the dimension of Z_T equals the number of vertices in T minus 1.

The set \tilde{Z} described above corresponds to the Σ_q -orbit of the Fuks cell Z corresponding to the tree T_Z with only one branching at level 2 and labels on the bottom level consistent with the left-right direction. In other words, the set $F = \{p_1, \dots, p_q\}$ with

$$p_i = (x_i^1, \dots, x_i^d)$$

is in Z if and only if

$$x_1^j = x_2^j = \dots = x_q^j$$

for every $j = 2, \dots, d$ and

$$x_1^1 < x_2^1 < \dots < x_q^1.$$

Note that the cell $\pi(Z)$ is the unique open cell of minimal dimension $d + q - 1$ of the Fuks partition of $B_q(\mathbb{R}^d)$.

In order to prove that $e(\alpha_q)^{d-1}$ is nonzero we have to prove that the homology class of $\pi(Z)$ is nontrivial in the compact support homology $H_{d+q-1}^c(B_q(\mathbb{R}^d); \pm\mathbb{Z})$. We must use the compact support homology (homology of the one-point compactification) because the manifolds $F_q(\mathbb{R}^d)$ and $B_q(\mathbb{R}^d)$ are open and the Poincaré–Lefschetz duality takes cohomology to the compact support homology. We always need twisted coefficients because for even d the manifold $B_q(\mathbb{R}^d)$ is oriented and $e(\alpha_q)^{d-1}$ is in the cohomology with twisted coefficients, while for odd d the orientation sheaf of $B_q(\mathbb{R}^d)$ is $\pm\mathbb{Z}$ and $e(\alpha_q)^{d-1}$ is in the untwisted cohomology.

We have to check that $\pi(Z)$ is not annihilated by the boundary map. The cells of the Fuks partition of dimension $d + q$ correspond to the trees T_1, \dots, T_{q-1} such that T_k has a binary branching $v \rightarrow w_a, w_b$ on level 3, then w_a has k children on level 1 and w_b has $q - k$ children on level 1. Each tree T_k corresponds to the Σ_q -orbit of the cell Y_k given by the (in)equalities:

$$x_1^j = x_2^j = \dots = x_q^j$$

for $j = 3, \dots, d$,

$$y_a = x_1^2 = x_2^2 = \dots = x_k^2 < x_{k+1}^2 = \dots = x_q^2 = y_b,$$

$$x_1^1 < \dots < x_k^1 \quad \text{and} \quad x_{k+1}^1 < \dots < x_q^1.$$

Now it remains to calculate the coefficient at $\pi(Z)$ in $\partial\pi(Y_k)$ (with appropriate coefficient twist). In [26] this coefficient was shown to be $\binom{q}{k}$ up to sign for $d = 2$. The calculations in [26] are actually applicable to the case $d > 2$ because the coordinates $j = 3, \dots, d$ are the same for all points in Y_k and Z and do not affect anything. Since the proof of [26, Theorem 2.5.1] is very brief and not very clear we present the calculations below.

For any $\rho \in \Sigma_q$ in order to make correct calculation we have to orient ρZ so that the map $\rho : Z \rightarrow \rho Z$ preserves the orientation. Note that this orientation coincides with the orientation given by the form $dx_*^d \wedge \dots \wedge dx_*^3 \wedge dx_*^2 \wedge dx_1^1 \wedge \dots \wedge dx_q^1$ (x_*^j denotes the common value of x_i^j) if and only if ρ is an even permutation.

Let us orient Y_k by the form $dx_*^d \wedge \dots \wedge dx_*^3 \wedge dy_a \wedge dy_b \wedge dx_1^1 \wedge \dots \wedge dx_q^1$. The boundary ∂Y_k corresponds to approaching the equality $x_*^2 = y_a = y_b$ from $y_a < y_b$ and it is therefore oriented by $dx_*^d \wedge \dots \wedge dx_*^3 \wedge dx_*^2 \wedge dx_1^1 \wedge \dots \wedge dx_q^1$. Denoting by $(-1)^\sigma$ the sign of a permutation σ we obtain:

$$\partial Y_k = \sum_{\sigma \in M_{k,q-k}} (-1)^\sigma \sigma Z,$$

where the subset $M_{k,q-k} \subset \Sigma_q$ consists of permutations σ such that

$$\sigma(1) < \dots < \sigma(k) \quad \text{and} \quad \sigma(k+1) < \dots < \sigma(q).$$

Note that $|M_{k,q-k}| = \binom{q}{k}$. For the homology with twisted coefficients we have to calculate:

$$(3.4) \quad \partial \sum_{\tau \in \Sigma_q} (-1)^\tau \tau Y_k = \sum_{\tau \in \Sigma_q, \sigma \in M_{k,q-k}} (-1)^\tau (-1)^\sigma \tau \sigma Z = \binom{q}{k} \sum_{\rho = \tau \sigma \in \Sigma_q} (-1)^\rho \rho Z.$$

Since q is a power of a prime p it follows that we have the congruence of polynomials in t :

$$(1+t)^q \equiv 1+t^q \pmod{p}$$

and therefore all the binomial coefficients $\binom{q}{k}$ are divisible by p . Hence all the coefficients of the boundary operator at $\sum_{\rho \in \Sigma_q} (-1)^\rho \rho Z$ are divisible by p and $\pi(Z)$ (which is actually the equivariant cycle $\sum_{\rho \in \Sigma_q} (-1)^\rho \rho Z$ in the homology with twisted coefficients) does represent a nonzero homology mod p .

Remark 3.1. Note the important thing: In (3.4) if we used the untwisted \mathbb{Z} coefficients and the corresponding cycle $\sum_{\tau \in \Sigma_q} \tau Y_k$ without signs then the resulting expression would be different and not divisible by $\binom{q}{k}$.

4. PROOF OF THEOREMS 1.4 AND 1.5

The proof follows the proof of Theorem 1.3, but with the following modifications. Let L be the $(n+1)$ -dimensional space of (non-homogeneous) linear functions on \mathbb{R}^n , or the space of homogeneous linear functions on \mathbb{R}^{n+1} restricted to S^n .

For $i = 1, \dots, n-1$, we define the maps

$$f_i : F_q(L) \rightarrow \alpha_q$$

as follows. For $F \in F_q(L)$ and $P(F) = \{V_1, \dots, V_q\}$ put

$$m_i(F) = \frac{1}{q} \sum_{j=1}^q \varphi_i(V_j(F)),$$

and

$$f_i : F \mapsto (\varphi_i(V_1(F)), \dots, \varphi_i(V_q(F))) - (m_i(F), \dots, m_i(F)).$$

Define the map f_n as before

$$f_n : F \mapsto \left(\mu(V_1) - \frac{1}{q} \mu(C), \dots, \mu(V_q) - \frac{1}{q} \mu(C) \right).$$

Note that the maps f_1, \dots, f_{n-1} are defined only for F such that all the sets $\{V_j(F)\}_{j=1}^q$ (we assume $V_j(F) = V_j(F) \cap C$) are nonempty. Moreover, these maps may be discontinuous. To correct this, consider the closed subset $Z \subseteq F_q(L)$ consisting of configurations F such that $f_n(F) = 0$. For $F \in Z$ the sets $V_j(F)$ have equal measures, and therefore they are convex compact sets with nonempty interiors (convex bodies), and they depend continuously (in the Hausdorff metric) on F , because their facets depend continuously on F . Now assume that the maps $f_1, \dots, f_{n-1} : Z \rightarrow \alpha_q$ are defined according to the above formulas; and extend each map f_i ($1 \leq i \leq n-1$) separately to a continuous Σ_q -equivariant map $f_i : F_q(L) \rightarrow \alpha_q$. This can be done because we extend them from a closed subspace and the target space is the Euclidean space.

Now we can use the Euler class and find a common zero of the maps f_1, \dots, f_n , i.e. the zero of

$$f_1 \oplus \dots \oplus f_n : F_q(L) \rightarrow (\alpha_q)^n.$$

The condition $f_n(F) = 0$ guarantees that $F \in Z$. That is we are in the range where the maps f_1, \dots, f_{n-1} are defined originally and the result follows.

5. PROOF OF THEOREM 1.6

Define the maps ($i = 1, \dots, k$)

$$f_i : F \mapsto \left(\mu_i(V_1) - \frac{1}{q} \mu_i(C), \dots, \mu_i(V_q) - \frac{1}{q} \mu_i(C) \right).$$

they are continuous on the whole $F_q(L)$. Again, let $Z \subset F_q(L)$ consist of configurations F such that $f_i(F) = 0$ for all $i = 1, \dots, k$. For $F \in Z$ the sets $V_1(F), \dots, V_q(F)$ are nonempty and have nonempty interior; of course, we assume $V_j(F) = V_j(F) \cap C$.

Now the maps ($i = k + 1, \dots, n$)

$$f_i : F \mapsto \left(\sigma_{i-k}(V_1 \cap \partial C) - \frac{1}{q} \sigma_{i-k}(\partial C), \dots, \sigma_{i-k}(V_q \cap \partial C) - \frac{1}{q} \sigma_{i-k}(\partial C) \right)$$

are defined on Z . Note that for $F \in Z$ (and in some neighborhood of $U \supset Z$) any two convex sets $V_j(F), V_l(F)$ are separated by a hyperplane $u_j(x) = u_l(x)$; and since $V_j(F)$ and $V_l(F)$ have nonempty interiors this hyperplane is transversal to ∂C . Therefore the sets $V_j(F) \cap \partial C$ depend continuously on $F \in U$ and the rest of the proof for a prime power q is similar to the previous proof.

Thus the case when q is a prime power is done. If q is not a prime power, we may iterate partitions in this theorem.

6. PROOF OF THEOREM 1.8

Again, the proof follows the proof of Theorem 1.3 with certain modifications.

The first $n - k$ maps $f_i : F_q(L) \rightarrow \alpha_q$ are given as before, by the measures μ_1, \dots, μ_{n-k} of the parts $P(F)$. The last k maps

$$f_{n-k+i} : F_q(L) \rightarrow \alpha_q$$

are given as follows: for $F \in F_q(L)$ and $P(F) = \{V_1(F), \dots, V_q(F)\}$, consider the coordinate function x_i in the target space of h , and put

$$m_i(F) = \frac{1}{q} \sum_{j=1}^q x_i(h(c(V_j(F)))).$$

Then define

$$f_{n-k+i} : F \mapsto (x_i(h(c(V_1(F)))), \dots, x_i(h(c(V_q(F))))) - (m_i(F), \dots, m_i(F)).$$

The maps f_{n-k+1}, \dots, f_n are defined only for F such that all $V_i(F)$ are nonempty. Since the first $n - k$ conditions

$$f_1(F) = \dots = f_{n-k}(F) = 0$$

define a closed subset $Z \subseteq F_q(L)$ and guarantee that all $V_i(F)$ have nonempty interiors, we can extend the maps f_{n-k+1}, \dots, f_n from Z continuously and Σ_q -equivariantly to the whole $F_q(L)$, and then apply Lemma 2.1 as above.

7. MEASURES ON THE SEGMENT AND THE COMPLEXITY OF THE MAXIMUM OF SEVERAL FUNCTIONS

Recall the “splitting necklace” theorem in its continuous version.

Theorem 7.1 (Noga Alon [3]). *Suppose we are given absolutely continuous probability measures μ_1, \dots, μ_n on a segment $[0, 1]$. For an integer $r \geq 2$ put $N = n(r - 1) + 1$. Then $[0, 1]$ can be partitioned into N segments I_1, \dots, I_N , the family $\mathcal{F} = \{I_i\}_{i=1, \dots, N}$ can be partitioned into r subfamilies $\mathcal{F}_1, \dots, \mathcal{F}_r$ so that for any $i = 1, \dots, n$ and $j = 1, \dots, r$*

$$\mu_i \left(\bigcup \mathcal{F}_j \right) = \frac{1}{r}.$$

Let us try to reduce Theorem 7.1 to Theorem 1.3.

Take L to be the set of polynomials of degree $\leq n$ on the segment $[0, 1]$. In this case we obtain q polynomials, the sets of the partition $P(F)$ are unions of several segments, and we have to show that the total number of segments does not exceed $n(q - 1) + 1$. This would follow from the following claim.

False Conjecture 7.2. *Suppose f_1, \dots, f_q are polynomials of degree $\leq n$, for $x \in \mathbb{R}$ denote*

$$g(x) = \max\{f_1(x), \dots, f_q(x)\}.$$

Then $g(x)$ has $\leq n(q - 1)$ points of switching between a pair of f_i 's.

Remark 7.3. The function $g(x)$ is usually called an *upper envelope* of the set of polynomials.

The case of non-prime-power r in the splitting necklace theorem would follow from this conjecture by iterating the splittings, as in the original proof of Theorem 7.1.

This conjecture is obviously true as stated for $n = 1$ or $q = 2$, the latter case gives Theorem 7.1 in case $r = 2^k$ by iterating (this is the same as using the “ham sandwich” theorem). The case $n = 2$ can also be done “by hand”, ordering the polynomials by the coefficient at x^2 and applying induction. But generally Conjecture 7.2 is false. Arseniy Akopyan has constructed a counterexample for $n = 3$, $q \geq 4$ (private communication). An unpublished result of P. Shor (cited in [1]) shows that for $n = 4$ the number of “switch” points may grow as $\Omega(q\alpha(q))$, where $\alpha(q)$ is the inverse Ackermann function. In [1] this problem was studied in a combinatorial setting. The sequence of “switches” between q polynomials may be encoded as a word in q letters with some restrictions depending on the degree n , such sequences are called *Davenport–Schinzel sequences*. It is known [1] that the maximum length of such a word complies with Conjecture 7.2 for $n = 1, 2$; but it is asymptotically superlinear in q for any fixed $n \geq 3$.

The following fact is known: Theorem 7.1 is tight and the number $n(r - 1) + 1$ cannot be made less. As a consequence, we obtain the following fact about analytic functions:

Theorem 7.4. *Suppose $L \subset C^\omega[0, 1]$ is an $(n + 1)$ -dimensional space of functions, q is a prime power. Then there exist distinct $f_1, \dots, f_q \in L$ such that the upper envelope*

$$g(x) = \max\{f_1(x), \dots, f_q(x)\}$$

has at least $n(q - 1)$ non-analytic points.

Proof. If for every subset $\{f_1, \dots, f_q\} \subset L$ the number of changes of maximum in $g(x)$ from $f_i(x)$ to $f_j(x)$ (they are exactly non-analytic points) is less than $n(q - 1)$, we would prove Theorem 7.1 using Theorem 1.3 with $< n(q - 1) + 1$ segments. But this is impossible. \square

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